A Fractional Viscoelastic Model of the Axon in Brain White Matter

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Introduction
Motivation:
Viscoelasticity is a sensitive measure of the composition of brain tissue and is an important marker in predicting neurodegenerative processes.

Objective:
- We aim to develop a numerical framework of the microstructure of white matter that incorporates a fractional viscoelastic constitutive material model of the composite white matter comprising of axons and glia.
- Representative volume elements (RVE) of axons embedded in glia with periodic boundary conditions are developed and subjected to a relaxation displacement boundary condition.
- Homogenized orthotropic fractional material properties of axons in glia are extracted by solving the inverse problem.

Finite Element Model of the Microstructure
Hexagonally packed RVEs of axons embedded in the ECM of varying volume fractions are developed with a periodic mesh using Abaqus FEA. Periodic boundary conditions (PBC) are enforced between nodes of opposite faces at any coordinate, x, and periodicity vector, p, by

\[ u_i(x_j + p_j) = u_i(x_j) + \frac{\partial u_i}{\partial x_j} p_j \]

where \( u_i \) is the displacement field in the \( x_i \) direction. In Abaqus, PBC can be implemented via a set of linear constraint equations with \( A_k \), number of coefficients. In the equation below, \( u'_i \) corresponds to displacement variable of node, \( i \) and degree of freedom, \( j \).

\[ A_1u'_1 + A_2u'_2 + \cdots + A_nu'_n = 0 \]

Finite viscoelasticity

The stress for a fractional viscoelastic springpot model is proportional to the \( j \)th order of strain

\[ \sigma(i) = C_p \phi^j(\epsilon(i)) \quad \forall (0 \leq \beta \leq 1) \]

A generalization of derivatives of non-integer orders can be obtained in a branch of mathematics called fractional calculus using the Liouville-Caputo derivative. The stress is thus

\[ \sigma(i) = \frac{k}{(1-\beta)} \int_0^t (t-\tau)^{\beta-1} \phi(i)(\tau) d\tau \]

For a 3D finite element model, writing \( k \) in terms of the volumetric and deviatoric components, the stress equation is

\[ \sigma(i) = \frac{1}{(1-\beta)} \int_0^t (K_0 - 2\beta G_0) \phi(i)(\tau) d\tau + \frac{1}{(1-\beta)} \int_0^t (C_0 (\phi(i)(\tau) + \phi'(i)(\tau)) d\tau \]

For a 3D state of stress, the Liouville-Caputo derivative can be numerically computed using the Grunwald-Letnikov operator

\[ \phi^{(j)}(i) = \frac{1}{\Gamma(1-\beta)} \sum_{k=0}^{M} \frac{1}{\Gamma(k+1-\beta)} \sigma(i) \phi((k+1-\beta)) + 2C_p \left( \frac{1}{\Gamma(1-\beta)} \sum_{k=0}^{M} \frac{1}{\Gamma(k+1-\beta)} \phi((k+1-\beta)) \right) \]

where \( k = \text{Total time} \)

\[ \phi^{(j)}(i) = \frac{(k - 1 - \beta)}{k} \phi(i) \]

Using the Grunwald-Letnikov operator, the deviatoric derivative of \( \phi \) is calculated:

\[ \frac{d\phi}{dt} = \frac{1}{\Gamma(1-\beta)} \sum_{j=1}^{M} \phi^{(j)}(k+1-\beta) \]

Material constants for Axon and ECM obtained by minimizing the logistic regression function and simulation for Acknowledgements:

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